MATH 4030 Problem Set 4[1]
Due date: Nov 5, 2019

Reading assignment: do Carmo's Section 3.2, 3.3, 4.2

Problems: (Those marked with † are optional.)

1. Calculate the mean curvature H and Gauss curvature K of the following surfaces:
$$S_1 = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\},$$
$$S_2 = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\},$$
with respect to the "upward" (toward positive z-axis) pointing unit normal N. Express the second fundamental form A of each surface at $p = (0, 0, 0)$ as a diagonal matrix. What are the principal curvatures and principal directions? Sketch the surfaces near $(0, 0, 0)$.

2. Compute the mean curvature H and Gauss curvature K of the *catenoid* given by the parametrization
$$X(u, v) = (\cosh v \cos u, \cosh v \sin u, v), \qquad (u, v) \in (0, 2\pi) \times \mathbb{R}.$$

3. Determine all the umbilic points on the ellipsoid (where $a > b > c > 0$ are distinct positive real numbers):
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
At which point(s) does the Gauss curvature K attains its maximum? What about for the mean curvature H (assuming the orientation is chosen such that $H > 0$)?

4. Let $S_1 = \{z = 0\}$ be the xy-plane and $S_2 = \{x^2 + y^2 = 1\}$ be the right unit cylinder. Show that the map $f : S_1 \to S_2$ defined by $f(x, y, 0) = (\cos x, \sin x, y)$ is a local isometry.

5. Let $f : S_1 \to S_2$ be an isometry between two compact surfaces S_1, S_2 in $\mathbb{R}^3$. Show that S_1 and S_2 have the same area. *(You can assume that S_2 is covered by a single parametrization except a set of measure zero.)*

6. (†) Show that a graphical surface $S = \{z = f(x, y)\}$ is *minimal* (i.e. $H \equiv 0$) if and only if f satisfies the *minimal surface equation*:
$$(1 + f_x^2)f_{yy} - 2f_x f_y f_{xy} + (1 + f_y^2)f_{xx} = 0.$$

7. (†) Find a local isometry $f : S_1 \to S_2$ from the upper half plane $S_1 = \{z = 0, y > 0\}$ to the cone $S_2 := \{x^2 + y^2 = z^2, z > 0\}$. Calculate the mean and Gauss curvatures of S_2.

8. (†) Given a surface $S \subset \mathbb{R}^3$, prove that the set of isometries $f : S \to S$ form a group under composition. This is called the *isometry group of S*. What is the isometry group of the unit sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$?

[1]Last revised on October 21, 2019